New Entanglement Witnesses and Entangled States

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Outline

- Why do we still need new entangled states / positive maps?

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- Positive maps via the "method of prescribing zeros"

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- Positive maps via the "method of prescribing zeros"
- SDP algorithm

Artwork by Sandbox Studio, Chicago with Ana Kova

’spooky action at a distance’

Ryan and Bryan meet Hilbert and Banach[1]

Hilbert spaces $\mathcal{H}$ $\longrightarrow$ operator algebras $B(\mathcal{H})$

[1] G. Aubrun and S. J. Szarek, *Alice and Bob Meet Banach: The interface of asymptotic geometric analysis and quantum information theory.* AMS (2017)

Ryan and Bryan meet Hilbert and Banach[1]

$$\begin{array}{lccc} & \text{Hilbert spaces } \mathcal{H} & \longrightarrow & \text{operator algebras } B(\mathcal{H}) \\ & \downarrow & \searrow^{\langle\cdot,\cdot\rangle_{\text{HS}}} & \downarrow \\ \text{today:} & \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} & & \text{matrix algebra } \mathrm{M}_{d_1 d_2} = B(\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}) \\ & \mathbb{C}^d \text{ or } \mathbb{R}^d & & \cup \end{array}$$

[1] G. Aubrun and S. J. Szarek, *Alice and Bob Meet Banach: The interface of asymptotic geometric analysis and quantum information theory.* AMS (2017)

Ryan and Bryan meet Hilbert and Banach[1]

$$
\begin{array}{llcl}
 & \text{Hilbert spaces } \mathcal{H} & \longrightarrow & \text{operator algebras } B(\mathcal{H}) \\
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 & \mathbb{C}^d \text{ or } \mathbb{R}^d & & \cup \\
 & & & \text{Hermitian matrices } \mathrm{M}^{\mathsf{sa}}_{d_1 d_2}
\end{array}
$$

[1] G. Aubrun and S. J. Szarek, *Alice and Bob Meet Banach: The interface of asymptotic geometric analysis and quantum information theory.* AMS (2017)

Ryan and Bryan meet Hilbert and Banach[1]

Hilbert spaces $\mathcal{H}$ $\longrightarrow$ operator algebras $B(\mathcal{H})$

$\downarrow$ $\searrow^{\langle\cdot,\cdot\rangle_{\mathsf{HS}}}$ $\downarrow$

today: $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$ matrix algebra $\mathrm{M}_{d_1 d_2} = B(\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2})$

$\mathbb{C}^d$ or $\mathbb{R}^d$ $\cup$

Hermitian matrices $\mathrm{M}^{\mathsf{sa}}_{d_1 d_2}$

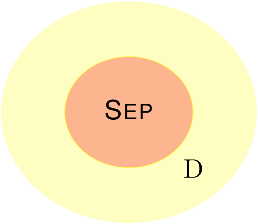
$\cup$

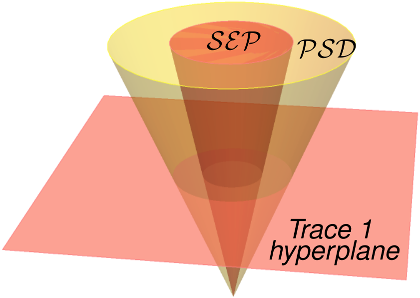
positive semidefinite $\mathcal{PSD}(\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2})$

[1] G. Aubrun and S. J. Szarek, *Alice and Bob Meet Banach: The interface of asymptotic geometric analysis and quantum information theory.* AMS (2017)

$\mathcal{SEP}\left(\mathbb{C}^{d_1}\otimes\mathbb{C}^{d_2}\right)\subset\mathcal{PSD}\left(\mathbb{C}^{d_1}\otimes\mathbb{C}^{d_2}\right)$

$\mathrm{M}^{\text{sa}}_{d_1 d_2}$: $\mathbb{R}$–vector space of dim $(d_1 d_2)^2$





$$\mathcal{SEP}\left(\mathbb{C}^{d_1}\otimes\mathbb{C}^{d_2}\right) := \text{conv}\left\{\mathcal{PSD}(\mathbb{C}^{d_1})\otimes\mathcal{PSD}(\mathbb{C}^{d_2})\right\}$$

Example: $\mathrm{M}_6^{\mathsf{sa}} \leftrightarrow \mathrm{M}_2^{\mathsf{sa}} \otimes \mathrm{M}_3^{\mathsf{sa}}$

$$\begin{pmatrix} 1 & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} =$$

$$\begin{pmatrix} 1 & \cdot \\ \cdot & \cdot \end{pmatrix} \otimes \begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} + \begin{pmatrix} \cdot & \cdot \\ \cdot & 1 \end{pmatrix} \otimes \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$$

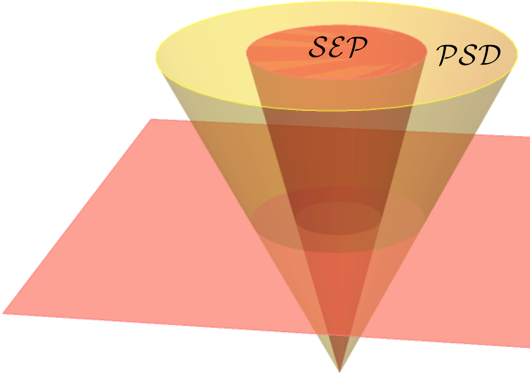
$$\pm \frac{1}{2} \begin{pmatrix} 1 & \pm 1 \\ \pm 1 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & \pm 1 & \cdot \\ \pm 1 & 1 & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$$

The separability problem

Given a positive semidefinite matrix

$$\rho \in \mathcal{PSD}\left(\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}\right)$$

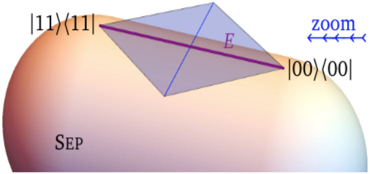
can you certify whether it is separable?



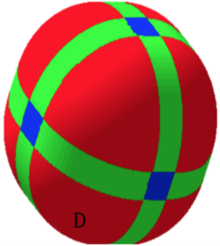
The separability problem is NP-hard[2]

[2]S. Gharibian, *Strong np-hardness of the quantum separability problem.* QIC (2010)

$\text{SEP}\left(\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}\right) \subset \mathrm{D}\left(\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}\right)$ inside $\mathrm{M}^{\mathrm{sa}}_{d_1 d_2}$



↙ Krein-Milman Thm



- Compact convex set D is much larger than SEP[3]
- D and SEP have the same inradius[4] w.r.t. HS norm and center $\frac{1}{d_1 d_2} I$

[3] I. Klep et al., *There are many more positive maps than completely positive maps.* IMRN (2019)
[4] L. Gurvits and H. Barnum, *Balls around maximally mixed bipartite quantum state.* Phys. Rev. A (2002)

Horodecki's entanglement witness theorem

A state ρ *on* $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$ *is entangled if and only if there exists a positive map* $\Phi\colon \mathrm{M}_{d_1}^{sa} \to \mathrm{M}_{d_2}^{sa}$ *such that the matrix* $\left(\Phi \otimes \mathrm{Id}_{\mathrm{M}_{d_2}^{sa}}\right)\rho$ *is not positive semidefinite.*[5]

For $\Phi = T$, the transposition, we get:

PPT criterion or Peres-Horodecki criterion

$$\mathcal{SEP} \subset \mathcal{PSD} \cap \Gamma(\mathcal{PSD}), \ \textit{where}\ \Gamma := T \otimes \mathrm{Id} \ \textit{(partial transpose)}$$

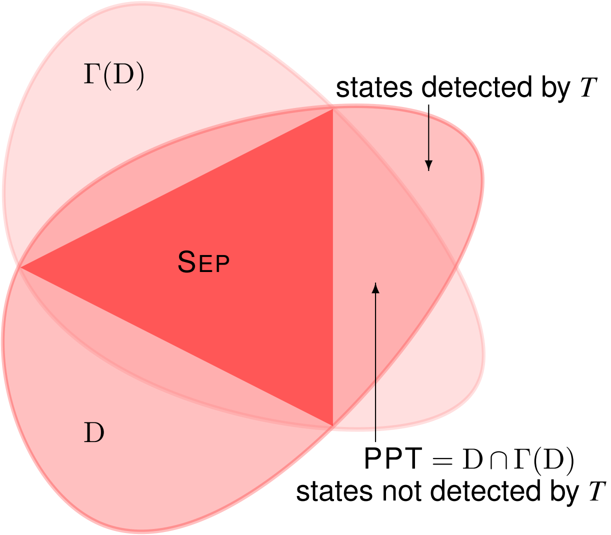
The strength of the PPT criterion is in detecting entanglement:

- If the partial transpose of a state is not positive, the state itself must be non-separable, i.e., entangled

[5]M. P. R. Horodecki, *Separability of mixed states: necessary and sufficient conditions.* Phys. Lett. A (1996)

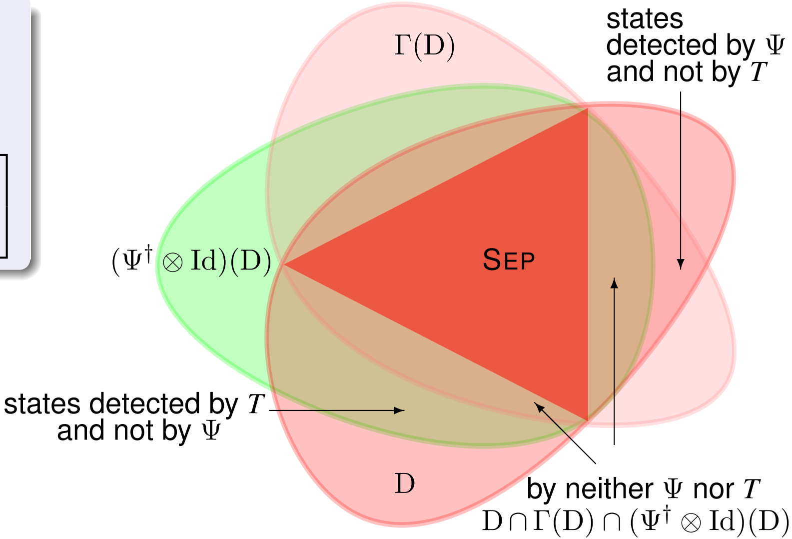
$\textsc{Sep} \subset \mathrm{D} \cap \Gamma(\mathrm{D})$, where $\Gamma = T \otimes \mathrm{Id}$

- Partial transposition detects entanglement in any pure state
- $\textsc{Sep}\left(\mathbb{C}^3 \otimes \mathbb{C}^3\right) \subsetneq \mathrm{PPT}\left(\mathbb{C}^3 \otimes \mathbb{C}^3\right)$



Choi map Ψ :

$$\begin{bmatrix} z_{00} & z_{01} & z_{02} \\ z_{10} & z_{11} & z_{12} \\ z_{20} & z_{21} & z_{22} \end{bmatrix} \downarrow \begin{bmatrix} z_{00}+z_{11} & -z_{01} & -z_{02} \\ -z_{10} & z_{11}+z_{22} & -z_{12} \\ -z_{20} & -z_{21} & z_{00}+z_{22} \end{bmatrix}$$



Choi isomorphism[6]

$$\begin{array}{ccc} B\left(\mathrm{M}_3,\ \mathrm{M}_3\right) & \xrightarrow{\mathit{Choi}} & B\left(\mathbb{C}^3 \otimes \mathbb{C}^3\right) \\ \Phi:\ \mathrm{M}_3 \to \mathrm{M}_3 & \mapsto & \mathit{Choi}(\Phi): \mathbb{C}^3 \otimes \mathbb{C}^3 \to \mathbb{C}^3 \otimes \mathbb{C}^3 \\ & & \| \\ & & \sum_{i,j} \Phi(E_{ij}) \otimes E_{ij} \end{array}$$

Choi matrix of Φ:

$$\mathit{Choi}(\Phi) = (\Phi \otimes \mathrm{Id})\ (|\chi\rangle\langle\chi|) \quad \text{where} \quad \chi = \sum_i e_i \otimes e_i.$$

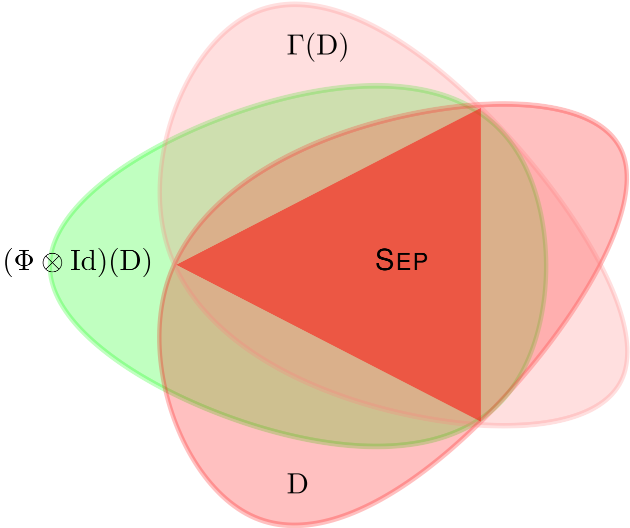
[6]Choi isomorphism vs. Jamiołkowski isomorphism: $\mathit{Choi} = \Gamma \circ \mathit{Jami}$

$$\Phi \in \boldsymbol{C}(\mathrm{M}_3, \mathrm{M}_3) \iff \mathit{Choi}(\Phi) \in \mathcal{C}(\mathbb{C}^3 \otimes \mathbb{C}^3)$$

Cone of superoperators $\boldsymbol{C}$		Cone of matrices $\mathcal{C}$		Dual cone $\mathcal{C}^*$
positive	$\boldsymbol{P}$	block positive	$\mathcal{BP}$	$\mathcal{SEP}$
	$\cup$		$\cup$	$\cap$
decomposable	$\boldsymbol{DEC}$	decomposable	co-$\mathcal{PSD} + \mathcal{PSD}$	$\mathcal{PPT}$
	$\cup$		$\cup$	$\cap$
completely positive	$\boldsymbol{CP}$	positive semidefinite	$\mathcal{PSD}$	$\mathcal{PSD}$
	$\cup$		$\cup$	$\cap$
PPT-inducing	$\boldsymbol{PPT}$	PPT	$\mathcal{PPT}$	co-$\mathcal{PSD} + \mathcal{PSD}$
	$\cup$		$\cup$	$\cap$
entanglement breaking	$\boldsymbol{EB}$	separable	$\mathcal{SEP}$	$\mathcal{BP}$

Positive maps $\Phi\colon \mathrm{M}_3^{\mathrm{sa}} \to \mathrm{M}_3^{\mathrm{sa}}$

$$\mathcal{SEP} \subset \bigcap_{\Phi} (\Phi \otimes \mathrm{Id})(\mathcal{PSD})$$



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What did Choi[7] do?

$\mathrm{x} = (x_0, x_1, x_2) \in \mathbb{R}^3$
$\mathrm{y} = (y_0, y_1, y_2) \in \mathbb{R}^3$

$\Phi\colon\ M_3^{\text{sym}} \to M_3^{\text{sym}}$	$p_\Phi(\mathrm{x},\mathrm{y}) := \langle \mathrm{y}\| \Phi(\|\mathrm{x}\rangle\langle \mathrm{x}\|) \|\mathrm{y}\rangle$
linear maps $\cup$ positive maps $\cup$ completely positive maps	biquadratic forms $\cup$ nonnegative biquadratic forms $\cup$ sums of squares (SOS)

[7] M.-D. Choi, *Positive semidefinite biquadratic forms.* LAA (1975)

Positive Choi map:

$$\Psi : \quad \mathrm{M}_3^{\mathsf{sym}} \to \mathrm{M}_3^{\mathsf{sym}}$$
$$\begin{bmatrix} z_{00} & z_{01} & z_{02} \\ z_{01} & z_{11} & z_{12} \\ z_{02} & z_{12} & z_{22} \end{bmatrix} \mapsto \begin{bmatrix} z_{00}+z_{11} & -z_{01} & -z_{02} \\ -z_{01} & z_{11}+z_{22} & -z_{12} \\ -z_{02} & -z_{12} & z_{00}+z_{22} \end{bmatrix}$$

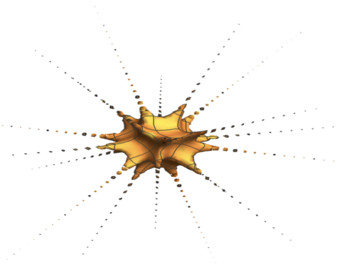
Nonegative biquadratic form $\langle y|\,\Psi\left(|x\rangle\langle x|\right)|y\rangle$:

$$x_0^2y_0^2 + x_1^2y_1^2 + x_2^2y_2^2 + x_0^2y_2^2 + x_1^2y_0^2 + x_2^2y_1^2 - 2x_0x_1y_0y_1 - 2x_0x_2y_0y_2 - 2x_1x_2y_1y_2$$

Positive Choi map:

$$\Psi : \quad M_3^{\text{sym}} \rightarrow M_3^{\text{sym}}$$

$$\begin{bmatrix} z_{00} & z_{01} & z_{02} \\ z_{01} & z_{11} & z_{12} \\ z_{02} & z_{12} & z_{22} \end{bmatrix} \mapsto \begin{bmatrix} z_{00}+z_{11} & -z_{01} & -z_{02} \\ -z_{01} & z_{11}+z_{22} & -z_{12} \\ -z_{02} & -z_{12} & z_{00}+z_{22} \end{bmatrix}$$



Nonegative biquadratic form $\langle y | \Psi (|x\rangle\langle x|) | y\rangle$:

$$x_0^2y_0^2 + x_1^2y_1^2 + x_2^2y_2^2 + x_0^2y_2^2 + x_1^2y_0^2 + x_2^2y_1^2 - 2x_0x_1y_0y_1 - 2x_0x_2y_0y_2 - 2x_1x_2y_1y_2$$

7 zeros: $(1,1,1;1,1,1)$, $(1,1,-1;1,1,-1)$, $(1,-1,1;1,-1,1)$, $(-1,1,1;-1,1,1)$, $(1,0,0;0,1,0)$, $(0,1,0,0,0,1)$, $(0,0,1;1,0,0)$

Number of zeros[8]

- Nonnegative biquadratic form which is not a sum of squares can have at most 10 zeros
- The number of real zeros of an SOS form is either infinite or at most 6

$\Rightarrow$ Nonnegative biquadratic forms with 7, 8, 9 or 10 zeros define positive maps that are not completely positive

[8] R. Quarez, *On the real zeros of positive semidefinite biquadratic forms. Commun. Algebra (2015)*

$\mathrm{x} = (x_0, x_1, x_2) \in \mathbb{C}^3$
$\mathrm{y} = (y_0, y_1, y_2) \in \mathbb{C}^3$

$\Phi : \ \mathrm{M}_3^{\mathsf{sa}} \to \mathrm{M}_3^{\mathsf{sa}}$	$p_\Phi(\mathrm{x}, \mathrm{y}) := \langle \mathrm{y} \vert \, \Phi\left(\vert \mathrm{x} \rangle\langle \mathrm{x} \vert \right) \vert \mathrm{y} \rangle$
positive maps	nonnegative forms

The zero set of Φ :

$$\{(\mathrm{x}, \mathrm{y}) \in \mathbb{C}^3 \times \mathbb{C}^3 \ : \ p_\Phi(\mathrm{x}, \mathrm{y}) = 0\}$$

Goal

Construct nonnegative polynomials $p_\Phi(\mathrm{x}, \mathrm{y})$*, which have 8, 9 or 10 real zeros.*

"10 zeros"

Zeros in $\mathbb{R}$:

$$(1,1,1;1,1,1), (1,1,-1;1,1,-1), (1,-1,1;1,-1,1), (-1,1,1;-1,1,1),$$
$$(1,t,0;t,1,0), (0,1,t;0,t,1), (t,0,1;1,0,t),$$
$$(1,-t,0;-t,1,0), (0,1,-t;0,-t,1), (-t,0,1;1,0,-t)$$

Zeros in $\mathbb{C}$:

$$(e^{i\varphi_0}, e^{i\varphi_1}, e^{i\varphi_2}; e^{i\varphi_0}, e^{i\varphi_1}, e^{i\varphi_2}),$$
$$(1, te^{i\varphi}, 0; te^{-i\varphi}, 1, 0), (0, 1, te^{i\varphi}; 0, te^{-i\varphi}, 1), (te^{i\varphi}, 0, 1; 1, 0, te^{-i\varphi})$$

Theorem ("10 zeros")

Superoperators $\Phi_t : \mathrm{M}_3^{sa} \to \mathrm{M}_3^{sa}$ *are positive for* $t \in \mathbb{R}$*:*

$$\begin{bmatrix} (t^2-1)^2 z_{00}+z_{11}+t^4 z_{22} & -(t^4-t^2+1)\, z_{01} & -(t^4-t^2+1)\, z_{02} \\ -(t^4-t^2+1)\, z_{10} & t^4 z_{00}+(t^2-1)^2 z_{11}+z_{22} & -(t^4-t^2+1)\, z_{12} \\ -(t^4-t^2+1)\, z_{20} & -(t^4-t^2+1)\, z_{21} & z_{00}+t^4 z_{11}+(t^2-1)^2 z_{22} \end{bmatrix}$$

Apart from $t = \pm 1$*, these positive maps are not completely or co-completely positive. Moreover,* Φ_t *define extreme rays in the cone of positive maps.*

"9 zeros"

Zeros in $\mathbb{R}$:

$$(1,1,1;1,1,1),(1,1,-1;1,1,-1),(1,-1,1;1,-1,1),(-1,1,1;-1,1,1),$$
$$(1,p,0;q,1,0),(1,-p,0;-q,1,0),$$
$$(0,1,q;0,p,1),(0,1,-q;0,-p,1),(0,0,1;1,0,0)$$

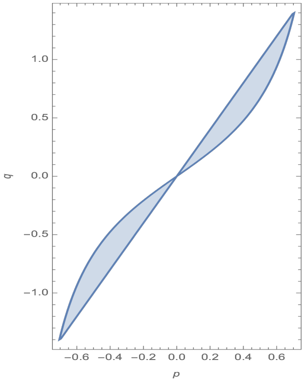
Zeros in $\mathbb{C}$:

$$(e^{i\varphi_0},e^{i\varphi_1},e^{i\varphi_2};\ e^{i\varphi_0},e^{i\varphi_1},e^{i\varphi_2}),$$
$$(1,p\,e^{i\varphi},0;\ q\,e^{-i\varphi},1,0),(0,1,q\,e^{i\varphi};\ 0,p\,e^{-i\varphi},1),(0,0,1;\ 1,0,0)$$

Theorem ("9 zeros")

$$\begin{bmatrix} D_{00} & -pq(1-q^2+p^2q^2)z_{01} & (pq-1)(p^2+pq-p^3q-q^2+p^2q^2)z_{02} \\ -pq(1-q^2+p^2q^2)z_{10} & D_{11} & -pq(1-q^2+p^2q^2)z_{12} \\ (pq-1)(p^2+pq-p^3q-q^2+p^2q^2)z_{20} & -pq(1-q^2+p^2q^2)z_{21} & D_{22} \end{bmatrix}$$

Positive, extremal and neither CP nor co-CP on $\mathcal{R}$



$(p, q) \in \mathcal{R}$

"8 zeros"

Zeros in $\mathbb{R}$:

$$(1,1,1;1,1,1),(1,1,-1;1,1,-1),(1,-1,1;1,-1,1),(-1,1,1;-1,1,1),$$
$$(1,0,0;\ m,1,0),\ (1,n,0;\ 0,1,0),\ (0,1,0;\ 0,0,1),\ (0,0,1;\ 1,0,0)$$

Zeros in $\mathbb{C}$:

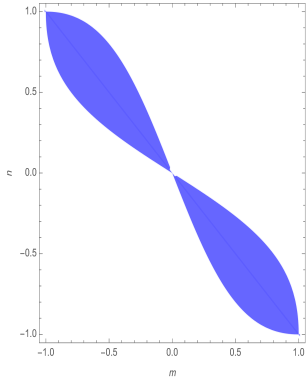
$$(1,1,e^{i\varphi};\ 1,1,e^{i\varphi}),\ (1,-1,e^{i\varphi};\ 1,-1,e^{i\varphi}),$$
$$(1,0,0;\ m,1,0),\ (1,n,0;\ 0,1,0),\ (0,1,0;\ 0,0,1),\ (0,0,1;\ 1,0,0)$$

Theorem ("8 zeros")

$$\begin{bmatrix} n^2\left(z_{00}+m(z_{01}+z_{10})+m^2 z_{11}\right) & -mn(nz_{00}-z_{01}+mnz_{10}-mz_{11}) & -n(m+n)(z_{02}+mz_{12}) \\ -mn(nz_{00}-z_{10}+mnz_{01}-mz_{11}) & m^2\left(n^2 z_{00}-n(z_{01}+z_{10})+z_{11}\right) & m(m+n)(nz_{02}-z_{12}) \\ -n(m+n)(z_{20}+mz_{21}) & m(m+n)(nz_{20}-z_{21}) & (m+n)^2 z_{22} \end{bmatrix}$$

$$+b\begin{bmatrix} z_{11} & 0 & -z_{02} \\ 0 & z_{22} & -z_{12} \\ -z_{20} & -z_{21} & z_{00}+z_{22} \end{bmatrix} + c\begin{bmatrix} 0 & z_{01}-z_{10} & 0 \\ z_{10}-z_{01} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Positive, extremal and neither CP nor co-CP on $\mathcal{A}$



$(m, n) \in \mathcal{A}$

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- Positive maps via the "method of prescribing zeros"
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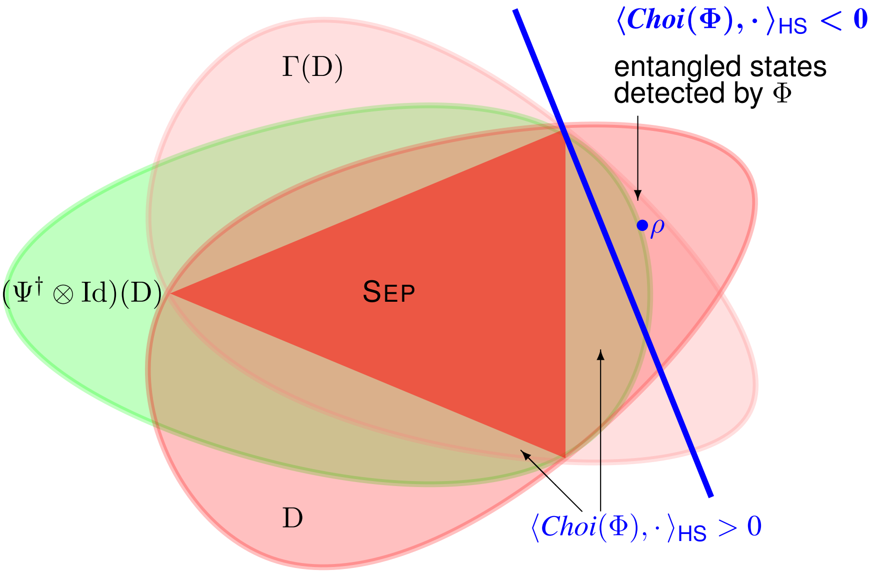


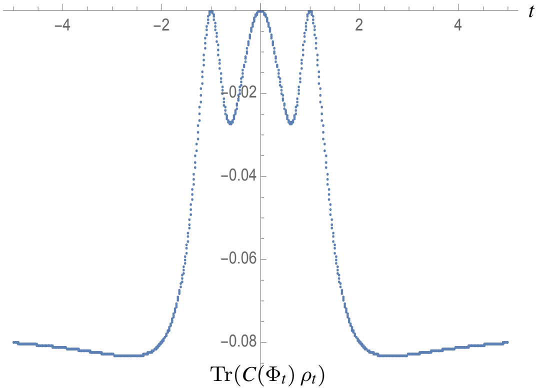
$\langle \boldsymbol{Choi}(\boldsymbol{\Phi}), \cdot \rangle_{\mathsf{HS}} < \mathbf{0}$
entangled states
detected by Φ
$\Gamma(\mathrm{D})$
ρ
$(\Psi^\dagger \otimes \mathrm{Id})(\mathrm{D})$
SEP
D
$\langle Choi(\Phi), \cdot \rangle_{\mathsf{HS}} > 0$

Algorithm: Semidefinite program

$$
\begin{aligned}
\textbf{minimize:} \quad & \operatorname{Tr}\left(Choi(\Phi)\, \rho \right) \\
\textbf{subject to:} \quad & (\Psi^{\dagger} \otimes \mathrm{Id})\rho \succeq 0 \\
& (T \otimes \mathrm{Id})\rho \succeq 0 \\
& \rho \succeq 0
\end{aligned}
$$

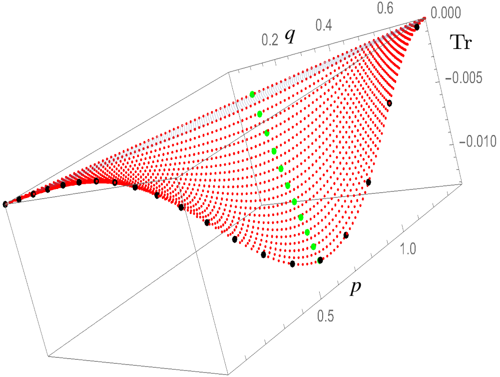
"10 zeros"

$$\rho_t = \left[\begin{array}{ccc|ccc|ccc} s_{00} & \cdot & \cdot & \cdot & s_{04} & \cdot & \cdot & \cdot & s_{04} \\ \cdot & s_{11} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & s_{22} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & s_{22} & \cdot & \cdot & \cdot & \cdot & \cdot \\ s_{04} & \cdot & \cdot & \cdot & s_{00} & \cdot & \cdot & \cdot & s_{04} \\ \cdot & \cdot & \cdot & \cdot & \cdot & s_{11} & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & s_{11} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & s_{22} & \cdot \\ s_{04} & \cdot & \cdot & \cdot & s_{04} & \cdot & \cdot & \cdot & s_{00} \end{array}\right]$$



"9 zeros"

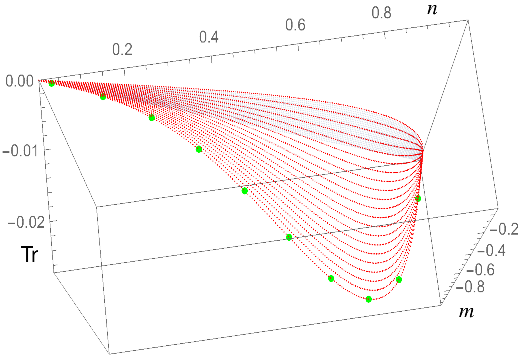
$$\rho_{p,q} = \left[\begin{array}{ccc|ccc|ccc}
s_{00} & \cdot & \cdot & \cdot & s_{04} & \cdot & \cdot & \cdot & s_{08} \\
\cdot & s_{11} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & s_{22} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & s_{33} & \cdot & \cdot & \cdot & \cdot & \cdot \\
s_{04} & \cdot & \cdot & \cdot & s_{44} & \cdot & \cdot & \cdot & s_{48} \\
\cdot & \cdot & \cdot & \cdot & \cdot & s_{55} & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & s_{66} & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & s_{77} & \cdot \\
s_{08} & \cdot & \cdot & \cdot & s_{48} & \cdot & \cdot & \cdot & s_{88}
\end{array}\right]$$



"8 zeros"

$$\rho_{m,n} =$$

$$\left[\begin{array}{ccc|ccc|ccc}
r_{00} & r_{01} & \cdot & r_{03} & r_{04} & \cdot & \cdot & \cdot & r_{08} \\
r_{01} & r_{11} & \cdot & r_{13} & r_{14} & \cdot & \cdot & \cdot & r_{18} \\
\cdot & \cdot & r_{22} & \cdot & \cdot & r_{25} & \cdot & \cdot & \cdot \\
\hline
r_{03} & r_{13} & \cdot & r_{33} & r_{34} & \cdot & \cdot & \cdot & r_{38} \\
r_{04} & r_{14} & \cdot & r_{34} & r_{44} & \cdot & \cdot & \cdot & r_{48} \\
\cdot & \cdot & r_{25} & \cdot & \cdot & r_{55} & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & r_{66} & r_{67} & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & r_{67} & r_{77} & \cdot \\
r_{08} & r_{18} & \cdot & r_{38} & r_{48} & \cdot & \cdot & \cdot & r_{88}
\end{array}\right]$$



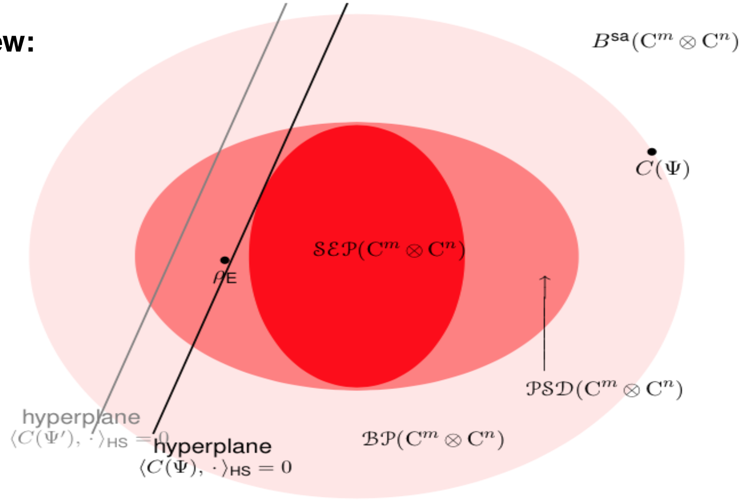
Conclusions

- New families of optimal entanglement witnesses
- A 5-parameter family of positive maps that amalgamates all the generalizations of Choi's map in the literature
- Extremality and non-CP come for free (from the number of zeros)

[9] A. Buckley and K. Šivic, *New examples of extremal positive linear maps, Linear Algebra Appl. (2020)*
[10] arXiv:2112.12643

Optimal Entanglement Witness

Bird's-eye view:



"Matrices" on bipartite Hilbert spaces

$d_1 d_2 \times d_1 d_2$ matrices

$$
\begin{array}{ccc}
B^{\mathsf{sa}}\left(\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}\right) & \equiv & \mathrm{M}^{\mathsf{sa}}_{d_1 d_2} \\
\updownarrow & & \updownarrow \\
B^{\mathsf{sa}}\left(\mathbb{C}^{d_1}\right) \otimes B^{\mathsf{sa}}\left(\mathbb{C}^{d_2}\right) & \equiv & \mathrm{M}^{\mathsf{sa}}_{d_1} \otimes \mathrm{M}^{\mathsf{sa}}_{d_2}
\end{array}
$$

$\mathcal{L}_{\mathbb{R}}$ {tensor products of $d_1 \times d_1$ and $d_2 \times d_2$ matrices}

Choi isomorphism

In specified bases,

$$
\begin{array}{ccc}
B\left(\mathrm{M}_n,\, \mathrm{M}_m\right) & \xrightarrow{C} & B\left(\mathbb{C}^m \otimes \mathbb{C}^n\right) \\
\Phi :\ \mathrm{M}_n \to M_m & \mapsto & C(\Phi) : \mathbb{C}^m \otimes \mathbb{C}^n \to \mathbb{C}^m \otimes \mathbb{C}^n, \\
 & & \Vert \\
 & & \sum_{i,j} \Phi(E_{ij}) \otimes E_{ij}
\end{array}
$$

Choi matrix of Φ:

$$C(\Phi) = (\Phi \otimes \mathrm{Id})\ (|\chi\rangle\langle\chi|)\,, \quad \chi = \sum_i e_i \otimes e_i.$$

Entanglement witness

For a state ρ on $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$, the following are equivalent:

1. state ρ is entangled,
2. there exists $\sigma \in \mathcal{SEP}^* = \mathcal{BP}$ such that $\langle \sigma, \rho \rangle_{\text{HS}} = \mathrm{Tr}(\sigma\rho) < 0$,
3. there exists a positive map $\Psi \colon \mathrm{M}_{d_2}^{\text{sa}} \to \mathrm{M}_{d_1}^{\text{sa}}$ such that $\mathrm{Tr}\,(C(\Psi)\rho) < 0$.

The Horodecki's entanglement witness theorem for a positive map Φ is a direct corollary of the above, where $\Phi = \Psi^\dagger$ from statement 3.

$\mathrm{x}, \mathrm{y} \in \mathbb{C}^3$

$\Phi\colon\ \mathrm{M}_3^{\mathsf{sa}} \to \mathrm{M}_3^{\mathsf{sa}}$	$p_\Phi(\mathrm{x},\mathrm{y}) := \langle \mathrm{y}\vert\, \Phi\left(\vert \mathrm{x}\rangle\langle \mathrm{x}\vert\right) \vert \mathrm{y}\rangle$
positive maps	nonnegative forms

Remark (The set of zeros.)

The group $PGL_3 \times PGL_3$ acts naturally on both, positive maps and nonnegative forms:

$$\begin{array}{ccc} \Psi(Z) & \mapsto & Q\,\Psi\left(PZP^*\right)Q^* \\ \langle \mathrm{y}\vert\,\Psi\left(\vert \mathrm{x}\rangle\langle \mathrm{x}\vert\right)\vert \mathrm{y}\rangle & \mapsto & \langle Q\,\mathrm{y}\vert\,\Psi\left(\vert P\,\mathrm{x}\rangle\langle P\,\mathrm{x}\vert\right)\vert Q\,\mathrm{y}\rangle \end{array}$$

"10 zeros"

For Ψ_t, we are minimizing

$$\mathrm{Tr}\left(C(\Psi_t)\,\rho\right) = \frac{1}{2\left(1-t^2+t^4\right)}\Big(s_{11}+s_{55}+s_{66}+t^4(s_{22}+s_{33}+s_{77})+ (1-t^2)^2(s_{00}+s_{44}+s_{88})- (1-t^2+t^4)(s_{04}+\overline{s_{04}}+s_{08}+\overline{s_{08}}+s_{48}+\overline{s_{48}})\Big).$$

Related work

- M.-D. Choi, Positive linear maps on C-algebras, Canad. Math. J. (1972)
- M.-D. Choi, Completely positive linear maps on complex matrices, Linear Algebra Appl. (1975)
- K.-C. Ha, Notes on extremality of the Choi map, Linear Algebra Appl. (2013)
- A. W. Harrow, A. Natarajan, and X. Wu, An improved semidefinite programming hierarchy for testing entanglement, Comm. Math. Phys. (2017)

Related work

- K.-C. Ha and S.-H. Kye, Entanglement witnesses arising from Choi type positive linear maps, J. Phys. A: Math. Theor. (2012)
- K.-C. Ha and S.-H. Kye, Exposedness of Choi-type entanglement witnesses and applications to lengths of separable states, Open Systems Information Dynamics (2013)
- K.-C. Ha and S.-H. Kye, Separable states with unique decompositions, Commun. Math. Phys. (2014)
- S.-H. Kye, Facial structures for various notions of positivity and applications to the theory of entanglement, Rev. Math. Phys. (2013).
- S.-H. Kye and H. Osaka, Classification of bi-qutrit positive partial transpose entangled edge states by their ranks, J. Math. Phys. (2012).